Solution of diffusion-advection equation of radon transport in many-layered geological media

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Abstract. Radon transport modeling in geological media is an important tool for solving problems and tasks of radioecology and geophysics. Comparison of radon field time series obtained by numerical and experimental methods is one of the most common and widely applicable ways to analyze the influence of state and variability of meteorological, electrical and actinometric parameters of atmosphere, cosmic weather factors, variations of deflected mode of geological medium on the level and variations of radon field. The solutions of stationary and non-stationary diffusion-advection equations of radon transport in many-layered geological media by numerical methods, notably by integro-interpolation method (balance method) are presented. The peculiarity of radon transport in many-layered media is taken into account in the developed numerical model. This peculiarity is connected with the transport equation coefficients which can change very rapidly at the border of two adjacent layers, i.e. they can be discontinuous at the borders of each layer that can be caused by parameters of soils greatly differing in value (density, porosity, radium content, diffusion and emanation coefficients). The present work is provided with an example of application of the developed numerical model for solving a practical problem on assessment of influence of deep seated uranium-containing rock on the value of radon volumetric activity at the depth of ≤ 1 m. The article considers non-stationary numerical model calculations showing at what time moments the distribution curves of radon volumetric activity coincide with stationary regime of radon transport in geological media. The validity of the developed numerical solution has been confirmed by these calculations.

Key words: radon • diffusion • advection • transport • modeling • soil

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Introduction

Radon transport modeling in geological media is widely applicable to solve practical problems and to explore the regularities of radon behavior [1–4, 6, 9, 10]. Based upon mathematical model the assessment of transport parameters is performed and predictions applicable in different spheres are made, for example: in building to amend the Code of Practice; in geological exploration to explore uranium ore deposits; in radioecology to assess radon danger for areas and buildings; in geophysics to study lithosphere-atmosphere bonds.

Radon transport modeling in geological media, which are described as similar to real ones, is a difficult issue, because geological medium is heterogeneous, layered with significantly different geophysical and geochemical characteristics for each layer. In this case a model of radon transport in layered medium with non-constant coefficients is required. Model coefficients can be the functions of spatial and time coordinates, and moreover, they can change very rapidly at a border of two adjacent layers, i.e. they can be discontinuous at the borders of each layer that can be caused by greatly
differing parameters of soils (density, porosity, radium content, diffusion and emanation coefficients).

In the explicit form there is no way to obtain analytical solution for such a model, therefore, in this case numerical methods are applicable.

The present work deals with obtaining numerical solution for stationary and non-stationary diffusion-advection equation for radon transport in many-layered geological media.

**Problem of stationary radon transport in many-layered medium with discontinuous coefficients**

Let us first consider the problem of stationary radon transport in many-layered medium with discontinuous coefficients, and then apply the modeling results to the case of non-stationary transport. As coefficients of transport equation can be discontinuous, then at the border of each layer the conditions of the so-called ideal contact (continuity) are specified, noting the equality of fluxes and volumetric activities of radon.

**Formulation of the problem**

Stationary radon transport in many-layered porous medium occurs by means of diffusion-advection mechanisms [5, 6]. It is required to assign distribution of radon volumetric activity along the depth of geological medium. According to such assignment the radon transport equation in medium for N-layered structure will have the following form (z-axis is directed downward from the ground surface, $ z \geq 0$).

$$
\frac{d}{dz} \left( D_n(z) \frac{dA_n(z)}{dz} \right) + \nu_n(z) \frac{dA_n(z)}{dz} - \lambda_n A_n(z) = 0
$$

Nonetheless, the following relationship (balance equation) expressing the conservation law for small volume [8]. The integrals and derivatives contained in balance equation should be substituted by approximate difference expressions. Thus, to solve the problem of radon transport in layered media we apply integro-interpolated method (balance method) of constructing conservative difference scheme.

Let us consider a calculation region of the problem (Eqs. (1), (2)). Let the thickness of geological medium $ l $ be a rather large value. Then, it can be expressed as the sum of appropriate layers: $ l = l_1 + l_2 + \ldots + l_{n-1} + l_n + l_{n+1} + \ldots + l_N $. Let us divide the layer thickness into $ K_l $ points, then $ l_i = i h_i $, where $ i = 0,1,2,\ldots,K_l-1 $, $ h_i $ - a grid pitch for layer $ i $. Similarly for layer $ l_2 $ we have $ l_2 = i h_2 $, where $ i_2 = K_{l_1}, K_{l_1} + 1, \ldots, K_{l_2} - 1 $. Carrying on the process for $ n $-layer we will obtain: $ l_n = h_n $, where $ i_n = K_{l_{n-1}}, K_{l_{n-1}} + 1, \ldots, K_{l_n} $. For simplicity, we can assume that $ h = h_1 = h_2 = \ldots = h_n = \ldots = h_{N_s} $, i.e. we will consider the uniform grid by spatial coordinate. Then, for the whole soil thickness the correlation will be performed: $ l = i h $, where $ i = 0,1,2,\ldots, K_l-1, K_l, K_l + 1, \ldots, K_{l_N} - 1, K_{l_N}, K_{l_N} + 1, \ldots, K_{l_N} - 1, K_{l_N} $. The region under consideration is represented schematically in Fig. 1.

For problem (Eq. (1), (2)) solution, the integro-interpolated method is applied. According to this method there occurred a transition from a system of differential equations (1) to algebraic system expressed in a difference form. Such transition occurs via some integral relation (balance equation) expressing the conservation law for small volume [8]. The integrals and derivatives contained in balance equation should be substituted by approximate difference expressions. Thus, to solve the problem of radon transport in layered media we apply integro-interpolated method (balance method) of constructing conservative difference scheme.

For Eq. (1) in $ n $-layer let us write down the balance equation for intercept $ z_{n+1/2} \leq z \leq z_{n+1/2} $ here $ z_{n+1/2} = h(i_n - \frac{1}{2}) $, $ h $ - pitch of difference scheme [8].

**Method of solution**

**Fig. 1.** The region under consideration for the problem (Eqs. (1), (2)).
Balance Eq. (3) reflects the conservation law for intercept \( z_{m+1/2} \leq z \leq z_{m+1/2} \). In order to obtain difference equation from balance Eq. (3), it is necessary to use the approximations of grid functions. We seek the function of solution at integer nodes \( A(z), z = z_n \), and diffusion and advection fluxes – at half-integer nodes. Let us substitute expressions (4), (5) and (6) into Eq. (3) and write down their approximation according to work [8].

\[
q_{n,k+1/2} - q_{n,k-1/2} = \frac{1}{h} \left( \frac{D_{nk+1/2} A_{nk+1} - D_{nk-1/2} A_{nk-1}}{2} + \frac{A_{nk+1} - A_{nk-1}}{2} \right)
\]

(4)

Let us further perform approximation of the second integral, which reflects advective radon flux, by quadrature trapezium rule

\[
\int_{z_{m+1/2}}^{z_{m+1/2}} v(z) \frac{dA(z)}{dz} dz = \frac{v_{n,k+1/2}}{2} (A_{n,k+1} - A_{n,k-1}) + \frac{v_{n,k-1/2}}{2} (A_{n,k+1} - A_{n,k-1})
\]

(5)

Let us perform approximation of other equation terms (radon decay and formation) by functions whose values are sought at integer nodes of the grid

\[
\alpha_{n,k} = D_{nk+1/2} A_{nk+1} + D_{nk-1/2} A_{nk-1} - \frac{v_{n,k+1/2}}{2} (A_{n,k+1} - A_{n,k-1})
\]

(6)

\[
\gamma_{n,k} = \frac{D_{nk+1/2} A_{nk+1} + D_{nk-1/2} A_{nk-1} - v_{n,k+1/2} A_{n,k+1} + v_{n,k-1/2} A_{n,k-1}}{2h} + \lambda
\]

(7)

\[
\beta_{n,k} = \frac{D_{nk+1/2} A_{nk+1} - D_{nk-1/2} A_{nk-1}}{2h}
\]

We solve system (7) for \( n \)-layer by double sweep method

\[
A_{n,k} = p_{nk+1/2} A_{nk+1} + q_{nk+1/2}
\]

\[
i_n = K_{n+1} A_{n+1} + 1, K_{n+1} A_{n+1} + 2, ..., K_n - 2
\]

(8)

Finally, the solution of problem (1, 2) can be written as

\[
A_{n+1,k} = p_{nk+1/2} A_{nk+1} + q_{nk+1/2}
\]

\[
i_{n+1} = K_{n+2} A_{n+2} + 2, K_{n+2} A_{n+2} + 3, ..., K_{n+1} - 1, K_{n+1}
\]

(9)

Solution of non-stationary radon transport equation in many-layered geological medium

Modeling of non-stationary radon transport in geological media is an important tool for solution of problems in radioecology, geophysics and atmosphere physics. Modeling of radon field dynamics in different geological media with further analysis of the obtained time series by method of their comparison with the results of direct and indirect measurements will allow to specify the parameters of radon transport model.
Balance method is applicable for numerical analysis of the problem (Eqs. (11)–(13)). Let us consider a calculation region for this problem. Let us introduce uniform grids: \( \omega = \{ z_{i} = i h, i = 0,1,2, \ldots , K_{x} \} \), where \( K_{x} \) – rather a large number and \( \omega = \{ t = j \tau, j = 0,1,2, \ldots , T \} \) with \( \tau = T / K_{t} \) with \( \tau \) of intercept \( 0 \leq t \leq T \). They form overall uniform grid: \( \omega_{p} = \omega_{z} \times \omega_{t} = \{ (z_{i}, t_{j}), z_{i} = i h, 0 \leq t \leq K_{x}, t_{j} = j \tau, 0 \leq j \leq T \} \). Let us introduce a grid function of solution \( A(z_{i}, t_{j}) = A_{ij} \in \omega_{p} \), which will be considered at integer nodes of uniform grid. Diffusion and advection fluxes will be considered at half-integer nodes. According to methodology mentioned above, we obtain three-diagonal system of algebraic Eqs. (14), after integrating by \( \int_{z_{i-1/2}}^{z_{i+1/2}} dt \), we obtain the solution

\[
\begin{align*}
\frac{\partial A_{ij}}{\partial t} - \frac{\partial}{\partial z} \left( D_{ij} \frac{\partial A_{ij}}{\partial z} \right) &= -v_{ij} A_{ij} + \partial A_{ij} + \lambda A(z_{i}, t) - A_{ij} = 0 \\
\frac{\partial A_{ij}}{\partial z} &= 0 \\
\frac{\partial A_{ij}}{\partial z} - \frac{\partial}{\partial t} \left( D_{ij} \frac{\partial A_{ij}}{\partial t} \right) &= -v_{ij} A_{ij} + \partial A_{ij} + \lambda A(z_{i}, t) - A_{ij} = 0
\end{align*}
\]

(11)

At the external borders of many-layered medium the boundary conditions are also assigned

\[
\lim_{z \to -\infty} A(z, t) = 0, \lim_{z \to +\infty} A(z, t) = A_{N,\infty}
\]

and initial condition

\[
A_{ij}(z, 0) = A_{N,\infty}(z)
\]

Method of solution

Balance method is applicable for numerical analysis of the problem (Eqs. (11)–(13)). Let us consider a calculation region for this problem. Let us introduce uniform grids: \( \omega_{z} = \{ z_{i} = i h, i = 0,1,2, \ldots , K_{x} \} \), with \( h = 1 / K_{x} \) with \( h \) on the semi-infinite interval \( 0 \leq z < \infty \), where \( K_{x} \) – rather a large number and \( \omega_{t} = \{ t_{j} = j \tau, j = 0,1,2, \ldots , T \} \) with \( \tau = T / K_{t} \) with \( \tau \) of intercept \( 0 \leq t \leq T \). They form overall uniform grid: \( \omega_{p} = \omega_{z} \times \omega_{t} = \{ (z_{i}, t_{j}), z_{i} = i h, 0 \leq t \leq K_{x}, t_{j} = j \tau, 0 \leq j \leq T \} \), i.e. the calculation region represents a rectangle. Let us introduce a grid function of solution \( A(z_{i}, t_{j}) = A_{ij} \in \omega_{p} \), which will be considered at integer nodes of uniform grid. Diffusion and advection fluxes will be considered at half-integer nodes. According to methodology mentioned above, we obtain three-diagonal system of algebraic Eqs. (14), after integrating by \( \int_{z_{i-1/2}}^{z_{i+1/2}} dt \), we obtain the solution

\[
\begin{align*}
-\alpha_{ij} A_{ij}^{n+1} + \beta_{ij} A_{ij}^{n+1} - \beta_{ij} A_{ij}^{n} &= F_{ij} \\
\gamma_{ij} &= \frac{\tau (D_{n+1/2}^{(+)} + D_{n-1/2}^{(+)} )}{h^{2}} + \frac{(v_{n+1/2} - v_{n-1/2})}{2h} \\
&\quad + \lambda z_{i}, \\
F_{n,i} &= A_{n,i}^{n} + \lambda z_{i} A_{n,i}
\end{align*}
\]

(14)

The solution of the system (14) can be obtained by the double sweep method. Applying this methodology to other layers, we obtain the solution

\[
A_{n+1,i}^{n+1} = p_{n+1,i} A_{n+1,i}^{n+1} + q_{n+1,i}
\]

Based on the described numerical model of radon transport in non-uniform geological media, the algorithm was developed and program SimRaTran [7] was created. This program allows to model radon transport in porous media represented by several (up to 20) emanating layers with different parameters. It also allows to calculate the distribution of radon volumetric activity and radon flux density along the depth, and radon flux density from the ground surface into atmosphere.

The results of numerical modeling

Let us consider one of the practical problems to solve which we can apply the numerical model (Eq. (10)) described above. The assessment of influence of uranium-containing rocks deposited at a lower depth on the value of near-surface radon activity (at depth of measurement, Fig. 2. Radon volumetric activity distribution along the depth of geological medium with three layers: 1 – uniform medium, \( u = 10^{-6} \text{ m·s}^{-1} \); 2 – uniform medium, \( u = 3 \times 10^{-6} \text{ m·s}^{-1} \); 3 – uniform medium, \( u = 10^{-6} \text{ m·s}^{-1} \).
which is usually $\leq 1$ m) is of great practical concern. Let us perform the calculations of the function of radon volumetric activity distribution along the depth for the area whose geological structure is layered.

Let us consider a medium consisting of three layers with a thickness of $l_1 = 4$ m, $l_2 = 2$ m and $l_3 = 8$ m, whose parameters ($D_n, K_n, \rho_n, \eta_n$) are similar for analysis simplification. In the first and third layers $A_{Ra} = 30$ Bq·kg$^{-1}$. In the second layer the specific activity of radium, which is uranium decay product, was chosen $A_{Ra} = 1000$ Bq·kg$^{-1}$, that is typical of uranium-containing rocks.

Numerical calculations were performed at different values and direction of advective velocity, taken in the range of values obtained in work [11]. At positive values of velocity $\nu$, the advective flux is directed towards the ground surface and is summed with diffusion flux, increasing the total radon flux into the atmosphere. At negative values of velocity $\nu$, the advective flux is directed downward from the ground surface decreasing the total radon flux into the atmosphere, and, respectively, the value of radon volumetric activity near the ground surface decreases.

Figure 2 represents the dependence functions of radon volumetric activity upon the depth at positive and negative velocity of advection $|\nu| = 10^{-6}$ m·s$^{-1}$.

The function of volumetric activity distribution along

**Fig. 3.** Radon volumetric activity distribution along the depth of geological medium with four layers at the moments of time $t$. (a) Radon distributions with Fig. 3c $\nu = 4 \times 10^{-6}$ m·s$^{-1}$ at the moments of time $t$: 1$-t=10^{7}$ s; 2$-t=10^{6}$ s; 3$-t=2 \times 10^{5}$ s; 4$-t=3 \times 10^{3}$ s. (b) Radon distributions with $\nu = -4 \times 10^{-6}$ m·s$^{-1}$ at the moment of time $t = 10^{7}$ s. (c) Radon distributions with $\nu = -4 \times 10^{-6}$ m·s$^{-1}$ at the same moments of time $t$. (d) Radon distributions with $\nu = 4 \times 10^{-6}$ m·s$^{-1}$ at the moment of time $t = 10^{7}$ s.
the depth for the uniform medium is shown in the figure for
comparison, i.e. all three layers are represented by a
loam with a similar radium content $A_{Ra} = 30$ Bq·kg$^{-1}$,
the advective velocity is assigned as $10^{-6}$ m·s$^{-1}$.

Analysis of the functions of radon volumetric activity
distribution (Fig. 2) has shown that the influence of the
second highly active layer, deposited even at a shallow
depth of 4–6 m, can have no impact on the radon
measurement results under certain circumstances. At
the negative advection velocity, the radon volumetric
activity at a depth of 2 m or less does not exceed the
values observed in relatively uniform medium.

Figure 3 represents the distribution curves of radon
volumetric activity in layered medium ($n = 4$) at different
moments of time $t$. The thickness of layers is: $l_1 = 1$ m;
$l_2 = 2$ m; $l_3 = 3$ m; $l_4 = 4$ m. For analysis simplification,
similar physico-geological parameters for each layer were
taken, except radium content which was 90 Bq·kg$^{-1}$ for
the first layer, 4 Bq·kg$^{-1}$ – for the 2nd layer, 30 Bq·kg$^{-1}$ – for
the 3rd layer and 1000 Bq·kg$^{-1}$ – for the 4th layer.

Figure 3a represents the family of calculation distribu-
tion curves of radon volumetric activity in geological
medium with the advection velocity of $\nu = 4 \times 10^{-6}$ m·s$^{-1}$.
The figure shows that the radon volumetric activity at
different moments of time is:

- $A = 25$–$50$ kBq·m$^{-3}$ – at the border of the 1st and
  2nd layers;
- $A = 25$–$100$ kBq·m$^{-3}$ – at the border of the 2nd and
  3rd layers;
- $A = 400$–$450$ kBq·m$^{-3}$ – at the border of the 3rd and
  4th layers.

Figure 3c represents the family of calculation distribu-
tion curves of radon volumetric activity in the medium
with negative advection velocity $\nu = -4 \times 10^{-6}$ m·s$^{-1}$.
Radon volumetric activity has been decreased to the
value of:

- $A = 10$–$25$ kBq·m$^{-3}$ – at the border of the 1st and
  2nd layers;
- $A = 25$ kBq·m$^{-3}$ – at the border of the 2nd and 3rd
  layers;
- $A = 200$–$250$ kBq·m$^{-3}$ – at the border of the 3rd and
  4th layers.

This effect confirms the fact that total radon flux
decreases when advection velocity is negative.

Figures 3b and 3d represent the cases, when within a
rather long time interval ($t = 10^7$ s) the curves calculated
by Eq. (15) obtained from non-stationary model solution
coincide with stationary regime of radon transport
(Eq. (10)). This validates the developed numerical solu-
tion (15). Thus, if the measurements are performed at
those periods of time when the advective flux is di-
rected deep into the ground, then the erroneous results
of potential radon risk assessment of the examined
area can be obtained. And vice versa, at large advective
fluxes into the atmosphere, radon volumetric activity at
the depth of measurement (usually 1 m) can increase
by 5 times.

Conclusion

A numerical model of non-stationary diffusion-advec-
tion radon transport in many-layered geological media
has been developed and its solution has been presented.
The peculiarity of radon transport in geological medium
has been taken into account in the developed model.
This peculiarity is connected with the transport equation
coefficients which can be discontinuous at the borders
of each layer.

The validity of the developed numerical solution
has been confirmed by calculations data using non-
stationary numerical model. These data show that at
certain moments of time the radon distribution curves
calculated by Eq. (15) obtained from non-stationary
model solution coincide with stationary regime of radon
transport (Eq. (10)).

The analysis of numerical modeling results, while
taking into account that advection velocity can vary over
a wide range of values and change its sign [11], indicates
the necessity to perform integral measurements of ra-
don activity in the soil air for obtaining valid assessment
of potential radon risk of the examined area.

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